

The Artin-Rees Lemma

We can also generalize the blowup construction for modules:

Let M an R -module, $\mathcal{J} : M = M_0 \supset M_1 \supset \dots$ an I -filtration.

Then

$$B_{\mathcal{J}} M := M \oplus M_1 \oplus \dots$$

is a graded $B_{\mathcal{J}} R$ module.

Prop: Assume each M_i of \mathcal{J} is finitely generated. Then \mathcal{J} is I -stable $\iff B_{\mathcal{J}} M$ is a f.g. $B_{\mathcal{J}} R$ -module.

Pf: If $B_{\mathcal{J}} M$ is f.g., the generators must be contained in the first n terms, for some n . Replacing them w/ their homogeneous components, we see that $M_n \oplus M_{n+1} \oplus \dots$ is generated by M_n , so

$$M_{n+i} = I^i M_n$$

for $i \geq 0$, so \mathcal{J} is stable.

Conversely, if \mathcal{J} is stable, then $M_{n+i} = I^i M_n$, some n , all $i \geq 0$. Thus, $B_{\mathcal{J}} M$ is generated by the union of sets of generators for M_0, \dots, M_n . \square

We can now easily prove the Artin-Rees Lemma:

Artin-Rees Lemma: let R be Noetherian, $I \subseteq R$ an ideal and $M' \subset M$ f.g. R -modules. If

$$\mathcal{J} : M = M_0 \supset M_1 \supset \dots$$

is I -stable, then so is

$$\mathcal{J}' : M' \supset M' \cap M_1 \supset \dots$$

Pf: $B_I R$ is a f.g. R -algebra (since I is f.g.). Thus, by Hilbert basis, it's Noetherian.

$B_{\mathcal{J}} M$ is a f.g. $B_I R$ module (by the above prop), so $B_{\mathcal{J}} M$ is Noetherian. Thus $B_{\mathcal{J}'} M' \subseteq B_{\mathcal{J}} M$ is f.g., so \mathcal{J}' is I -stable. \square

Note that this means \exists some n s.t. $\forall i \geq n, I(M_i \cap M') = M_{i+1} \cap M'$.

The following important result comes as an easy corollary:

Krull intersection Theorem: let R be Noetherian, $I \subseteq R$ an ideal. If M is a finitely generated R -module, then there is $r \in I$ s.t.

$$(1-r) \left(\bigcap_{j=1}^{\infty} I^j M \right) = 0.$$

If R is a domain or a local ring, and I is a proper ideal, then

$$\bigcap_{j=1}^{\infty} I^j = 0.$$

The second statement can be interpreted geometrically. Roughly, if a function vanishes along I to arbitrarily high order, then it's 0. This may fail for a non integral domain:

Ex: $R = k[x] / (x^2 - x)$ \bullet \bullet
 $x=0$ $x=1$

Set $I = (x)$. Then $I^2 = (x^2) = (x)$, so $x \in \bigcap I^j$.

This is because x vanishes to infinite order on the two points, but it's not 0.

Pf of Krull intersection: Let $M' = \bigcap_{i=1}^{\infty} I^i M \subseteq M$. M is finitely generated over a Noetherian ring, so it's Noetherian. Thus M' is f.g., so we can apply Artin-Rees and set

$$M_i = I^i M.$$

Then $M' \cap M \supset M' \cap IM \supset \dots$ is I stable, so there's some integer p s.t.

$$M' \cap M_{p+1} = I(M' \cap M_p)$$

But $M' \subseteq M_{p+1} \subseteq M_p$ so $M' = IM'$.

We don't know I is in the Jacobson radical, so we can't apply Nakayama, but we can apply a corollary of Cayley-Hamilton and get $\exists r \in I$ s.t. $(1-r)M' = 0$

For the second statement, take $M=R$. Then $M' = \bigcap I^i$.

If R is a domain, since I is proper, $1-r \neq 0$, so $1-r$ is a nonzerodivisor, so $M'=0$.

If R is local, then $1-r$ is a unit, so $M'=0$. \square

From this, we get an easy corollary.

Cor: Let R be a Noetherian local ring, $I \subset R$ a proper ideal. If $\text{gr}_I R$ is a domain, so is R .

Pf: Suppose $f, g \in R$ and $fg=0$. Then $\text{in}(f)\text{in}(g)=0$ (see HW). So $\text{in}(f)$ or $\text{in}(g)$ is 0.

By Krull, $\bigcap I^n = 0$ so f or g is 0. \square